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Applications of Sturm sequences to bifurcation analysis of delay differential equation models

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Abstract

This paper formalizes a method used by several others in the analysis of biological models involving delay differential equations. In such a model, the characteristic equation about a steady state is transcendental. This paper shows that the analysis of the bifurcation due to the introduction of the delay term can be reduced to finding whether a related polynomial equation has simple positive real roots. After this result has been established, we utilize Sturm sequences to determine whether a polynomial equation has positive real roots. This work has extended the stability results found in previous papers and provides a novel theorem about stability switches for low degree characteristic equations.
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1. General method

When one introduces a time delay into a system of differential equations, it is often of interest to determine whether or not bifurcations occur for various lengths of the delay. In particular, a stable steady state can become unstable if, by increasing the length of the time delay, the eigenvalues of the system go from having negative real parts to having positive real parts, and this occurs only if they traverse the imaginary axis. Many authors

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have utilized certain methods for determining if and when a bifurcation occurs about a steady state [3,5,11,12,16]. We also suggest the reader to visit the book by Kuang [8] and Stepan [15] for a nice overview of general stability results. In this paper, we review these techniques but maintain the analysis in polynomial form. Once the standard polynomial results are developed, we show how these techniques can extend the analysis of a system of delay differential equations by Nelson and Perelson [12]. In the last section, we introduce Sturm sequences which provide a novel algorithm for determining stability of low degree, i.e., less than degree 4, polynomials that may be useful to anyone interested in stability analysis.

1.1. Existence of critical delays

The transcendental equation of the delayed differential equation, at the steady state determined for $\tau = 0$, will have the form

$$P(\lambda, \tau) \equiv P_1(\lambda) + P_2(\lambda)e^{-\lambda\tau} = 0, \tag{1}$$

where τ is the length of the discrete time delay, and P_1 and P_2 are polynomials in λ. We can rewrite (1) as

$$\sum_{j=0}^{N} a_j \lambda^j + e^{-\lambda\tau} \sum_{j=0}^{M} b_j \lambda^j = 0 \tag{2}$$

and assume that the steady state about which we have linearized is stable in the absence of the delay. Then for $\tau = 0$ all of the roots of the polynomial have negative real part. As τ varies, these roots change. We are interested in any critical values of τ at which a root of this equation transitions from having negative to having positive real parts. If this is to occur, there must be a boundary case, a critical value of τ, such that the characteristic equation has a purely imaginary root [4]. The following demonstrates how to determine whether or not such a τ exists, by reducing (1) to a polynomial problem and seeking particular types of roots, thus determining whether a bifurcation can occur as a result of the introduction of delay.

Early stability methods developed and presented in the classic papers of Pontryagin [13] and Nyquist [7] have been used for many years to study bifurcations in transcendental equations. However, these methods rely heavily on the principal of the argument for determining where the poles of the transcendental equations are located. In other words, they use geometric principles to determine the number of roots of these equations. The monograph by Chebotarev and Meiman [1] shows how to extend the Routh–Hurwitz criteria for polynomials to quasi-polynomials. However, it has been noted that the application of the Chebotarev criterion as an analytical tool is not effective practically [15]. The results that we present in this paper using Sturm sequences relax the need for the application of the argument principle and provides an analytical criterion that is practical to use.

We begin by looking for a purely imaginary root, $i\nu$, $\nu \in \mathbb{R}$, of (1),

$$P_1(i\nu) + P_2(i\nu)e^{-i\nu\tau} = 0. \tag{3}$$

We can then separate the polynomial into its real and imaginary parts, and write the exponential in terms of trigonometric functions to get

$$R_1(\nu) + iQ_1(\nu) + \big(R_2(\nu) + iQ_2(\nu)\big)\big(\cos(\nu\tau) - i\sin(\nu\tau)\big) = 0. \tag{4}$$

Note that because $i\nu$ is purely imaginary, R_1 and R_2 are even polynomials of ν, while Q_1 and Q_2 are odd polynomials. In order for (4) to hold, both the real and imaginary parts must be 0, so we get the pair of equations

$$\begin{aligned} &R_1(\nu) + R_2(\nu)\cos(\nu\tau) + Q_2(\nu)\sin(\nu\tau) = 0, \quad \text{and} \\ &Q_1(\nu) - R_2(\nu)\sin(\nu\tau) + Q_2(\nu)\cos(\nu\tau) = 0, \end{aligned} \tag{5}$$

Squaring each equation and summing the results yields

$$R_1(\nu)^2 + Q_1(\nu)^2 = R_2(\nu)^2 + Q_2(\nu)^2. \tag{6}$$

We notice two things about (6). First, this is a polynomial equation where the trigonometric terms have disappeared and the delay, τ, has been eliminated. Secondly, it is an equality of *even* polynomials.

Define a new variable $\mu = \nu^2$. Then Eq. (6) above can be written in terms of μ as

$$S(\mu) = 0, \tag{7}$$

where S is a polynomial in μ. Note that we are only interested in $\nu \in \mathbb{R}$, and thus if all of the real roots of S are negative, we will have shown that there can be no simultaneous solution ν^* of (5). Conversely, if there is a positive real root μ^* to S, there is a delay τ corresponding to $\nu^* = \pm\sqrt{\mu^*}$ which solve both equations in (5). This is easily seen from the complex form of (4).

An alternate approach, which follows along the lines of the D-partition method or the τ-decomposition method, on finding the roots of the characteristic equation (1) is taken in [9] and [10]. In this case, for $\lambda = i\nu$, we rewrite (1) as

$$-\frac{P_1(i\nu)}{P_2(i\nu)} = e^{-i\nu\tau}. \tag{8}$$

As ν varies, plotting the right-hand side in the complex plane traces out a unit circle, and the left-hand side is a rational curve. The intersections of these curves represent the critical delays in which we are interested. Thus finding the roots of the characteristic equation comes down to finding values of ν for which the left-hand side of (8) has modulus 1. This reproduces Eq. (6), and the freedom to choose τ again ensures that the original characteristic polynomial (1) is satisfied for some τ^*. However, again these methods are more geometric in their approach.

1.2. Nondegeneracy

Having found a critical delay τ^* and the point $z = i\nu^*$ at which a root of the characteristic equation hits the imaginary axis, it is necessary to confirm that the root continues into the positive half-plane as τ increases past τ^*. The following lemma reviews the necessary and sufficient conditions to guarantee this.

Lemma 1. *If* $\lambda = i\nu^*$ *and* $\tau = \tau^*$ *satisfy the characteristic equation* (1), *then*

$$\frac{d}{d\tau}\operatorname{Re}(\lambda)\bigg|_{\lambda=i\nu^*,\,\tau=\tau^*} > 0$$

if and only if

$$R_1(\nu^*)R_1'(\nu^*) + Q_1(\nu^*)Q_1'(\nu^*) \neq R_2(\nu^*)R_2'(\nu^*) + Q_2(\nu^*)Q_2'(\nu^*). \tag{9}$$

Proof. Beginning with the characteristic equation (1), we can write

$$e^{-\lambda\tau} = -\frac{P_1(\lambda)}{P_2(\lambda)} \to -\lambda\tau = \log\left(-\frac{P_1(\lambda)}{P_2(\lambda)}\right).$$

Note: we do not consider the degenerate cases where $P_1(i\nu^\star)$ or $P_2(i\nu^\star)$ are zero, as in these cases there exists a purely imaginary eigenvalue for all τ. Thus we may assume that $P_i(\lambda) \neq 0$ in a neighborhood of $\lambda = i\nu^\star$. Taking the derivative with respect to τ (treating λ as a function of τ, $\lambda = \lambda(\tau)$) gives

$$-\lambda - \tau\frac{d\lambda}{d\tau} = \frac{P_1'(\lambda)P_2(\lambda) - P_1(\lambda)P_2'(\lambda)}{P_1(\lambda)P_2(\lambda)} \cdot \frac{d\lambda}{d\tau},$$

where $' = \frac{d}{d\lambda}$. At $\lambda = i\nu^*$ and $\tau = \tau^*$, the left-hand side becomes $-i\nu^* - \tau^*\frac{d\lambda}{d\tau}$. Since $i\nu^*$ is purely imaginary, and τ^* is real, $\frac{d\lambda}{d\tau}$ is purely imaginary if and only if

$$\frac{P_1'(i\nu^*)P_2(i\nu^*) - P_1(i\nu^*)P_2'(i\nu^*)}{P_1(i\nu^*)P_2(i\nu^*)}$$

is real. This occurs only when the numerator and denominator are real multiples of one another. Now we can write

$$\begin{aligned}&\frac{P_1'(i\nu^*)P_2(i\nu^*) - P_1(i\nu^*)P_2'(i\nu^*)}{P_1(i\nu^*)P_2(i\nu^*)}\\ &\quad= \frac{(Q_1' - iR_1')(R_2 + iQ_2) - (Q_2' - iR_2')(R_1 + iQ_1)}{(R_1 + iQ_1)(R_2 + iQ_2)}.\end{aligned}$$

Collecting real and imaginary parts, we find that

$$\frac{d}{d\tau}\operatorname{Re}(\lambda)\bigg|_{\lambda=i\nu^*,\,\tau=\tau^*} = 0$$

if and only if

$$\frac{Q_1'R_2 + R_1'Q_2 - Q_2'R_1 - R_2'Q_1}{R_1R_2 - Q_1Q_2} = \frac{Q_1'Q_2 - R_1'R_2 + R_1R_2' - Q_1Q_2'}{R_1Q_2 + R_2Q_1}.$$

Cross multiplying and canceling like terms yields

$$R_1R_1'\left(R_2^2 + Q_2^2\right) + Q_1Q_1'\left(R_2^2 + Q_2^2\right) = R_2R_2'\left(R_1^2 + Q_1^2\right) + Q_2Q_2'\left(R_1^2 + Q_1^2\right).$$

But at $\nu = \nu^*$, $R_1^2 + Q_1^2 = R_2^2 + Q_2^2 \neq 0$. So this reduces to the condition

$$R_1R_1' + Q_1Q_1' = R_2R_2' + Q_2Q_2',$$

which is the necessary and sufficient condition for

$$\frac{d}{d\tau}\,\mathrm{Re}(\lambda)\bigg|_{\lambda=i\nu^*,\,\tau=\tau^*}=0.$$

Thus the derivative is not equal to 0 if (9) holds. □

Practically, this condition can be checked by formally differentiating Eq. (6) with respect to ν and verifying that equality does not hold for $\nu=\nu^*$. In summary, we have reduced the question of whether the introduction of a delay can cause a bifurcation to a problem of determining if a polynomial has any positive real roots. If such roots can be found, then the argument above guarantees that there is a delay size τ^* such that one of the eigenvalues of the system crosses the imaginary axis, destabilizing its critical point. We have proven the following lemma.

Lemma 2. *Given a system of differential equations $\dot{x}(t)=f(x(t),x(t-\tau))$ with a discrete delay τ, and a stable steady state for x_s for $\tau=0$, and let*

$$\sum_{i=1}^{N}a_i\lambda^i+e^{-\lambda\tau}\sum_{i=1}^{M}b_i\lambda^i=0$$

be the characteristic equation of the system about x_s. Then there exists a $\tau^*>0$ for which x_s undergoes a nondegenerate change of stability if and only if the equation

(i) $S(\mu)=0$ (*as defined in Eq.* (7)) *has a positive real root* $\mu^*=(\nu^*)^2$, *such that*
(ii) $S'(\mu^*)\neq 0$.

That is, when μ^* is a simple, positive real root of the equation (*simple because* $S'(\mu)\neq 0$)

$$\Big[\sum(-1)^j a_{2j}\mu^j\Big]^2+\mu\Big[\sum(-1)^j a_{2j+1}\mu^j\Big]^2$$
$$=\Big[\sum(-1)^j b_{2j}\mu^j\Big]^2+\mu\Big[\sum(-1)^j b_{2j+1}\mu^j\Big]^2.$$

2. Positive real roots and Sturm sequences

Once the polynomial equation (7) has been obtained, one must determine whether it has any positive real roots. There are many approaches one might take. For degree 2 characteristic polynomials, there is always the quadratic formula. For third and fourth degree polynomials, there are also explicit algorithms (see, for example, [6] or [10]).

One approach to showing that no bifurcation exists is to apply the Routh–Hurwitz condition. If these conditions are satisfied, then all of the roots of (7) have negative real part, and thus none are positive and real. This condition is not sharp, however, since there remains the possibility that the polynomial (7) has a conjugate pair of roots with positive real part and nonzero imaginary part. For example, consider the characteristic polynomial

$$\lambda^2+3\lambda+5+\lambda e^{-\lambda\tau}=0. \tag{10}$$

In the absence of delay, this becomes,

$$\lambda^2 + 4\lambda + 5 = 0,$$

which clearly has only roots with negative real part, and thus the steady state is stable. Explicitly, the roots are $\lambda_{1,2} = -2 \pm i$. The polynomial (7) produced by the process we have described is

$$\mu^2 - 2\mu + 25 = 0,$$

whose roots are $1 \pm 2i\sqrt{6}$. This polynomial has no positive real solution, and yet fails the Routh–Hurwitz conditions.

In other words, the Routh–Hurwitz conditions can guarantee the absence of a bifurcation, but cannot give conditions under which a bifurcation *does* occur with increasing τ.

A simple approach to determining whether a positive real root exists is Descartes' Rule of Signs, whereby the number of sign changes in the coefficients is equal to the number of positive real roots, modulo 2. If the number of sign changes is odd, then a solution is guaranteed. If, however, the number of sign changes is even, the rule cannot distinguish between, for example, 2 roots and 0 roots.

A more general approach to this problem is Sturm sequences. Suppose that a polynomial f has no repeated roots. Then f and f' are relatively prime. Let $f = f_0$ and $f' = f_1$. We obtain the following sequence of equations by the division algorithm:

$$\begin{aligned} f_0 &= q_0 f_1 - f_2, \\ f_1 &= q_1 f_2 - f_3, \\ &\vdots \\ f_{s-2} &= q_{s-2} f_{s-1} - K, \end{aligned}$$

where K is some constant.

The sequence of *Sturm functions*, $f_0, f_1, f_2, \ldots, f_{s-1}, f_s (= K)$ is called a *Sturm chain*. We may determine the number of real roots of the polynomial f in any interval in the following manner: plug in each endpoint of the interval, and obtain a sequence of signs. The number of real roots in the interval is the difference between the number of sign changes in the sequence at each endpoint. For a complete proof of the method of Sturm sequences, see [14].

Given a specified parameter set, this method gives a simple, implementable algorithm for determining whether a bifurcation occurs, without the need to run the full simulation of the system of equations for various delays.

3. Applications

In [12], we are faced with the characteristic equation

$$\lambda^3 + A\lambda^2 + \left(B - \delta c e^{-\lambda\tau}\right)\lambda + \delta c\rho - \delta c(\rho - \psi')e^{-\lambda\tau} = 0, \tag{11}$$

where $A \equiv \delta + c + \rho$, $B \equiv \delta c + (\delta + c)\rho$, and $\psi' \equiv \rho - d_T > 0$, the notation being that of the paper. In the paper, it is shown that for $\tau \ll 1$ and $\tau \gg 1$ no change of stability occurs. We can extend this result to all $\tau > 0$.

In the notation we have been using, Eq. (11) yields

$$R_1(\nu) = -A\nu^2 + \delta c\rho, \qquad Q_1(\nu) = -\nu^3 + B\nu,$$
$$R_2(\nu) = -\delta c d_T, \qquad Q_2(\nu) = -\delta c \nu.$$

Using these specific polynomials, (6) becomes

$$\nu^6 + \big(A^2 - 2B\big)\nu^4 + \big(B^2 - (\delta c)^2 - 2\delta c\rho A\big)\nu^2 - (\delta c)^2\big(\psi'^2 - 2\rho\psi'\big) = 0, \quad \text{or}$$
$$\mu^3 + \big(A^2 - 2B\big)\mu^2 + \big(B^2 - (\delta c)^2 - 2\delta c\rho A\big)\mu - (\delta c)^2\big(\psi'^2 - 2\rho\psi'\big) = 0. \tag{12}$$

This can be simplified by substituting the known values of A, B, and ψ'. For the μ^2 coefficient, we have

$$\begin{aligned} A^2 - 2B &= (\delta + c + \rho)^2 - 2\big(\delta c + (\delta + c)\rho\big) \\ &= \delta^2 + c^2 + \rho^2 + 2\delta c + 2\rho c + 2\delta\rho - 2\delta c - 2(\delta + c)\rho \\ &= \delta^2 + c^2 + \rho^2. \end{aligned}$$

Further, for the μ coefficient, we have

$$\begin{aligned} B^2 - 2\delta c\rho A - (\delta c)^2 &= \big((\delta c)^2 + (\delta\rho)^2 + (c\rho)^2 + 2\delta^2 c\rho + 2\delta\rho c^2 + 2\rho^2\delta c\big) \\ &\quad - 2\delta c\rho(\rho + c + \delta) - (\delta c)^2 \\ &= (\delta\rho)^2 + (c\rho)^2. \end{aligned}$$

And for the constant term we have

$$\psi^2 - 2\rho\psi' = \psi'(\rho - d_T - 2\rho) = -\psi'(\rho + d_T).$$

So we may write Eq. (12) as

$$\mu^3 + \big(\delta^2 + c^2 + \rho^2\big)\mu^2 + \big((\delta\rho)^2 + (c\rho)^2\big)\mu + (\delta c)^2\psi'(\rho + d_T) = 0.$$

This is a polynomial with positive coefficients, and cannot have any positive real roots, therefore the introduction of a delay into the model in Nelson and Perelson [12] cannot lead to a bifurcation for any length of the time delay. Nelson and Perelson only showed an asymptotic result for small and large time delays. Hence maintaining polynomial form in the study of stability switching of delay differential equations can in some cases strengthen the results that we find through asymptotic methods.

In [11], the following characteristic equation is encountered for a system of delay differential equations:

$$\lambda^2 + (\delta + c)\lambda + \delta c - \eta e^{-\lambda\tau} = 0,$$

where δ, c, and η are positive constants. We have $P_1(\lambda) = \lambda^2 + (\delta + c)\lambda + \delta c$, and $P_2(\lambda) = -\eta$. Thus

$$R_1(\nu) = -\nu^2 + \delta c, \qquad Q_1(\nu) = (\delta + c)\nu, \qquad R_2(\nu) = -\eta, \quad \text{and} \quad Q_2(\nu) = 0.$$

By the method of the lemma, we arrive at

$$\begin{aligned}
\eta^2 &= \left(\nu^2 - \delta c\right)^2 + (\delta + c)^2 \nu^2, \\
\eta^2 &= \nu^4 - 2\delta c \nu^2 + \delta^2 c^2 + \left(\delta^2 + 2\delta c + c^2\right)\nu^2, \\
0 &= \nu^4 + \left(\delta^2 + c^2\right)\nu^2 + \delta^2 c^2 - \eta^2.
\end{aligned} \tag{13}$$

Let $\mu = \nu^2$, then this becomes

$$S(\mu) \equiv \mu^2 + \left(\delta^2 + c^2\right)\mu + \delta^2 c^2 - \eta^2 = 0.$$

Since the linear coefficient of S is positive, by Descartes' rule of signs, a positive real root can occur if and only if the constant coefficient is negative. So a change of stability occurs if and only if $0 > \delta^2 c^2 - \eta^2 = (\delta c + \eta)(\delta c - \eta)$, i.e., if and only if $\delta c < \eta$.

Checking nondegeneracy, we take the derivative of the last line of (13), and check that equality does not hold,

$$0 = 4(\nu^*)^3 + 2\left(\delta^2 + c^2\right)\nu^* \quad \text{and} \quad 0 = 4(\nu^*)^2 + 2\left(\delta^2 + c^2\right),$$

which clearly has no roots. This shows, that a nondegenerate bifurcation does occur for $\delta c < \eta$. This reproduces the results in Nelson et al. [11].

Culshaw and Ruan [3] applied this same method to conclude that no bifurcations occurred in a delay model with characteristic equation

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 e^{-\lambda\tau} + a_4\lambda e^{-\lambda\tau} + a_5 = 0. \tag{14}$$

In their paper, Culshaw and Ruan follow the method we have presented in Lemma 2, and arrive at the polynomial S if Eq. (7) in the form

$$z^3 + \alpha z^2 + \beta z + \gamma.$$

Proposition 2 in [3] states that if $\gamma \geqslant 0$ and $\beta > 0$, then this polynomial has no positive real root. The proof of this proposition also assumes that $\alpha > 0$. In this case all of the coefficients are positive, and there are certainly no positive roots. The condition $\alpha, \beta, \gamma > 0$ is sufficient, but it is not necessary for no roots to exist. In the next section we develop a criterion which will extend this result and give necessary and sufficient conditions for a characteristic equation of the form (14) to produce no bifurcations.

4. General order two and three characteristic equations

Using Sturm sequences, we can derive some general results for low order characteristic equations. We begin with the general degree two equation, for which a general result is easy:

$$\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} = 0. \tag{15}$$

A steady state with this characteristic is stable for $\tau = 0$ if all of the roots of

$$\lambda^2 + (a + c)\lambda + (b + d) = 0$$

have negative real part. By the Routh–Hurwitz conditions, this occurs if and only if $a + c > 0$ and $b + d > 0$.

Letting $\lambda = i\nu$ and proceeding as in Lemma 2, we arrive at the following form for (7):

$$\mu^2 + \left(a^2 - c^2 - 2b\right)\mu + \left(b^2 - d^2\right) = 0. \tag{16}$$

Let $A \equiv a^2 - c^2 - 2b$ and $B \equiv b^2 - d^2$. (16) has a positive real root in two circumstances. Clearly, since the lead coefficient is positive, if $B < 0$, then there is a positive real root. If $B > 0$, the roots of (16) are

$$\frac{-A \pm \sqrt{A^2 - 4B}}{2},$$

and there is a simple positive root if and only if $A < 0$. Thus we can conclude with the following proposition.

Proposition 1. *A steady state with characteristic equation* (15) *is stable in the absence of delay, and becomes unstable with increasing delay if and only if*

(i) $a + c > 0$ *and* $b + d > 0$, *and*
(ii) *either* $b^2 < d^2$, *or* $b^2 > d^2$ *and* $a^2 < c^2 + 2b$.

For similar results in the degree two case, and also for some more general results, see Kuang [8].

For the degree three problem, the situation is somewhat more complex. The general characteristic equation is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + \left(b_2\lambda^2 + b_1\lambda + b_0\right)e^{-\lambda\tau} = 0. \tag{17}$$

The steady state is stable in the absence of delay if the roots of

$$\lambda^3 + (a_2 + b_2)\lambda^2 + (a_1 + b_1)\lambda + (a_0 + b_0) = 0$$

have negative real part. This occurs if and only if $a_2 + b_2 > 0$, $a_0 + b_0 > 0$, and $(a_2 + b_2)(a_1 + b_1) - (a_0 + b_0) > 0$.

In this case the form of (7) is

$$\mu^3 + A\mu^2 + B\mu + C = 0, \tag{18}$$

where

$$A \equiv a_2^2 - b_2^2 - 2a_1, \qquad B \equiv a_1^2 - b_1^2 + 2b_2b_0 - 2a_2a_0, \quad \text{and} \quad C \equiv a_0^2 - b_0^2. \tag{19}$$

As in the degree two case, since the lead coefficient is positive, there are two manners in which a positive real root can occur. The first and simplest is to have $C < 0$. Now suppose that $C > 0$. Since the polynomial is odd, we are guaranteed a negative real root. The only way to have a simple positive real root in this case is to have 2 positive real roots. In other words, all of the roots are real. Now suppose we take the Sturm chain of the polynomial (18), denoted f_0, f_1, f_2, f_3. We evaluate the entire real line, i.e., from $-\infty$ and ∞, and construct a table of the signs at these endpoints. $f_0 = \mu^3 + A\mu^2 + B\mu + C$ and $f_1 = 3\mu^2 + 2A\mu + B$, so we have

	$-\infty$	∞
f_0	$-$	$+$
f_1	$+$	$+$
f_2		
f_3		

We know that there must be three real roots. The difference in the number of sign changes at each endpoint must be three, but this is only possible if the Sturm sequence at one endpoint is always positive or always negative, and the sequence at the other endpoint must alternate. So the completed table must have the form

	$-\infty$	∞
f_0	$-$	$+$
f_1	$+$	$+$
f_2	$-$	$+$
f_3	$+$	$+$

Notice that f_0 and f_2 are odd degree polynomials, and f_1 and f_3 are even degree polynomials, and the signs at $-\infty$ are the direct consequence of those at ∞ (the same for even polynomials, and the opposite for odd polynomials). Thus, the bifurcation occurs in the case $C > 0$ if and only if the lead coefficients f_2 and f_3 are positive. Carrying out the division algorithm, the lead coefficient of f_2 is

$$-\left(\frac{2}{3}B - \frac{2}{9}A^2\right),$$

which is positive if and only if $A^2 - 3B > 0$. f_3 is the constant

$$-\frac{9}{4}\frac{4B^3 - A^2B^2 - 18ABC + 4CA^3 + 27C^2}{(A^2 - 3B)^2}.$$

After some algebraic manipulation, we can see that this is positive if and only if

$$4\bigl(B^2 - 3AC\bigr)\bigl(A^2 - 3B\bigr) - (9C - AB)^2 > 0. \tag{20}$$

Now we have conditions to guarantee that there are three real roots. We must finally guarantee that one of these is positive. This occurs if (18) has a positive critical point. The derivative function is

$$f_1 = 3\mu^2 + 2A\mu + B,$$

whose roots are $(-2A \pm 2\sqrt{A^2 - 3B})/6$. One of these is positive if $A < 0$ or $A > 0$ and $B < 0$, so either A or B must be negative. So we have the following theorem.

Theorem 1. *A steady state with characteristic equation* (17) *is stable in the absence of delay, and becomes unstable with increasing delay if and only if A, B, and C are not all positive and*

(i) $a_2 + b_2 > 0$, $a_0 + b_0 > 0$, $(a_2 + b_2)(a_1 + b_1) - (a_0 + b_0) > 0$, *and*

(ii) *either* $C < 0$, *or* $C > 0$, $A^2 - 3B > 0$ *and the condition* (20) *is satisfied, where* A, B, *and* C *are given by* (19).

Note that the condition where $C = 0$ allows (18) to have a zero eigenvalue and thus the linearization theory will fail. This could then be studied via Liapanov functions.

5. Conclusions

We have developed a method of reducing the question of the existence of a delay-induced loss of stability to the problem of finding real positive roots of a polynomial. Stepan [15] comments that there are a variety of methods for determining bifurcations of transcendental equations but none of these methods can be generally used for functional differential equations. Each method has its place in the field of use. For example, the widely used D-partition method depends heavily on the knowledge of the hypersurfaces, which are generally difficult to find [2]. Another example is the Yesipovich–Svirskii criterion which is a simpler version of the Pontryagin criterion but still relies heavily on geometry and the application of the argument principle [15]. The method that we presented using the Sturm sequences does not utilize the argument principle and thus simplifies the task of determining the necessary and sufficient conditions for the roots of a quasi-polynomial to have negative real parts. We also find this method to be very general through its application to biological problems.

One of the keys for this is maintaining the quasi-polynomial structure in the form of the polynomials involved. These results are summarized in Lemma 2. The method of this lemma can be used to verify and to extend the results in several cases from the literature. More generally, it is easy, using the technique, to arrive at a general conditions on the coefficients of a characteristic equation of degree 2, such that it describes an asymptotically stable steady state which becomes unstable as the delay parameter is increased. This simple, practical test is given in Proposition 1, and is related to analysis done by Kuang [8, Chapter 3].

Further, the method of Lemma 2 can be used to verify or to extend the scope of several examples in the literature. However, the main result of this paper, presented in Theorem 1, is for the degree three case, where Sturm sequences are used to develop an elementary (though perhaps algebraically complicated) test for bifurcation. It is hoped that this criterion will make the investigation of third order systems of delay differential equation simpler, both analytically and numerically. It provides a general algorithm for determining stability that anyone utilizing delay differential equation models can apply.

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